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Solvability of the F_4 Integrable System

Konstantin G. Boreskov¹,*Institute for Theoretical and Experimental Physics,
Moscow 117259, Russia***Juan Carlos Lopez V.²,***Instituto de Ciencias Nucleares, UNAM, A.P. 70-543,
04510 Mexico D.F., Mexico***Alexander V. Turbiner^{3,4}***Laboratoire de Physique Theorique, Université Paris-Sud, France and
Instituto de Ciencias Nucleares, UNAM, A.P. 70-543, 04510 México*

Abstract

It is shown that the F_4 rational and trigonometric integrable systems are exactly-solvable for *arbitrary* values of the coupling constants. Their spectra are found explicitly while eigenfunctions by pure algebraic means. For both systems new variables are introduced in which the Hamiltonian has an algebraic form being also (block)-triangular. These variables are invariant in respect to the Weyl group of F_4 root system and can be obtained by averaging over an orbit of the Weyl group. Alternative way of finding these variables exploiting a property of duality of the F_4 model is presented. It is demonstrated that in these variables the Hamiltonian of each model can be expressed as a quadratic polynomial in the generators of some infinite-dimensional Lie algebra of differential operators in a finite-dimensional representation. Both Hamiltonians preserve the same flag of polynomials and each subspace of the flag coincides with the finite-dimensional representation space of this algebra. Quasi-exactly-solvable generalization of the rational F_4 model depending on two continuous and one discrete parameters is found.

¹boreskov@vxitep.itep.ru

²vieyra@nuclecu.unam.mx

³turbiner@lyre.th.u-psud.fr, turbiner@nuclecu.unam.mx

⁴On leave of absence from the Institute for Theoretical and Experimental Physics,
Moscow 117259, Russia.

1 Introduction

The F_4 model was originally found in the Hamiltonian reduction method in ref.[1] (see also the review [2]). This model describes a quantum system in four-dimensional space with Hamiltonian which is associated with the root system R of the F_4 algebra [2]:

$$\begin{aligned} \mathcal{H}_{F_4}^{(OP)}(z) = & \frac{1}{2} \sum_{i=1}^4 (-\partial_{z_i}^2 + \omega^2 z_i^2) + g_1 \sum_{j>i} \{v(z_i - z_j) + v(z_i + z_j)\} \\ & + g \sum_{i=1}^4 v(z_i) + g \sum_{\nu' s=0,1} v\left(\frac{z_1 + (-1)^{\nu_2} z_2 + (-1)^{\nu_3} z_3 + (-1)^{\nu_4} z_4}{2}\right) . \end{aligned} \quad (1.1)$$

The arguments of a potential function v depend on the scalar products $(\alpha, z) = \sum_{i=1}^4 \alpha_i z_i$, and summation goes over all positive roots $\alpha \in R_+ = \{e_i, e_i \pm e_j, (e_1 \pm e_2 \pm e_3 \pm e_4)/2, 1 \leq i, j \leq 4\}$.

In the rational F_4 model the function v takes a form

$$v(z) = \frac{1}{z^2} , \quad (1.2)$$

and the corresponding Hamiltonian becomes

$$\begin{aligned} \mathcal{H}_{F_4}^{(OP,r)} = & \frac{1}{2} \sum_{i=1}^4 (-\partial_{z_i}^2 + \omega^2 z_i^2) + g_1 \sum_{j>i} \left(\frac{1}{(z_i - z_j)^2} + \frac{1}{(z_i + z_j)^2} \right) \\ & + g \sum_{i=1}^4 \frac{1}{z_i^2} + 4g \sum_{\nu' s=0,1} \frac{1}{[z_1 + (-1)^{\nu_2} z_2 + (-1)^{\nu_3} z_3 + (-1)^{\nu_4} z_4]^2} , \end{aligned} \quad (1.3)$$

where ω is the harmonic oscillator frequency and $g, g_1/4 > -1/8$ are the coupling constants. The configuration space is given by

$$-\infty < z_1 \leq z_2 \leq z_3 \leq z_4 < \infty . \quad (1.4)$$

and it coincides with the Weyl chamber.

In the case of the trigonometric F_4 model the oscillator term in (1.1) is absent, $\omega = 0$, and

$$v(z; \beta) = \frac{\beta^2}{\sin^2 \beta z} , \quad (1.5)$$

where β is a parameter. Hence, the Hamiltonian takes a form

$$\begin{aligned}
H_{F_4}^{(OP,t)} = & -\frac{1}{2} \sum_{i=1}^4 \partial_{z_i}^2 + g_1 \beta^2 \sum_{j>i} \left(\frac{1}{\sin^2 \beta(z_i - z_j)} + \frac{1}{\sin^2 \beta(z_i + z_j)} \right) \\
& + g \beta^2 \sum_{i=1}^4 \frac{1}{\sin^2 \beta z_i} + g \beta^2 \sum_{\nu'_s} \frac{1}{\sin^2 \beta \frac{[z_1 + (-1)^{\nu_2} z_2 + (-1)^{\nu_3} z_3 + (-1)^{\nu_4} z_4]}{2}} ,
\end{aligned} \tag{1.6}$$

with coupling constants $g, g_1/4 > -1/8$. The configuration space is given by the simplex

$$-\infty < z_1 \leq z_2 \leq z_3 \leq z_4 < \infty , \quad z_1 - z_4 < \frac{\pi}{\beta} \tag{1.7}$$

and it coincides with the Weyl alcove. When β tends to zero the trigonometric Hamiltonian degenerates to the rational one at $\omega = 0$. Both the rational and trigonometric F_4 models are completely-integrable for arbitrary coupling constants g, g_1 . If $g = 0$ the Hamiltonian (1.1) degenerates to the (rational or trigonometric) D_4 model.

Making a change of variables in (1.1)

$$z_{1,2} = x_1 \pm x_2 \quad , \quad z_{3,4} = x_3 \pm x_4 \tag{1.8}$$

we come to an equivalent Hamiltonian

$$\begin{aligned}
\mathcal{H}_{F_4}(x) = & \frac{1}{2} \sum_{i=1}^4 (-\partial_{x_i}^2 + 4\omega^2 x_i^2) + 2g \sum_{j>i} \{v(x_i - x_j; \beta) + v(x_i + x_j; \beta)\} \\
& + 2g_1 \sum_{i=1}^4 v(2x_i; \beta) + 2g_1 \sum_{\nu'_s=0,1} v[x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4; \beta] .
\end{aligned} \tag{1.9}$$

Similarly to (1.1) this form can be associated with the dual root system of the F_4 algebra $R_+^\vee = \{2e_i, e_i \pm e_j, (e_1 \pm e_2 \pm e_3 \pm e_4)\}$ and we refer to the substitution (1.8) as to the *duality* transformation. Due to relation $v(2x, \beta) = (1/4)v(x, 2\beta)$ one can see that both in the rational and trigonometric cases this transformation interchanges the coupling constants $g_1 \rightleftharpoons 2g$. Besides that in the trigonometric case it rescales one part of the potential (this corresponds to changing in this part $\beta \rightarrow 2\beta$ in potential functions $v(x; \beta)$) preserving the other part. In particular, if $g = 0$ in (1.1) the transformation (1.8) converts (1.1) into the Hamiltonian of D_4 problem.

It turns out that an analysis and formulas for the F_4 trigonometric problem are much simpler in x -coordinates than in z -coordinates originally used. Similar phenomenon was observed in G_2 model [8], where among two equivalent systems of relative coordinates (both of which were equally suitable for the rational model) the only one led to an algebraic form

of the trigonometric Hamiltonian ⁵. In what follows we shall always use the form (1.9) of the Hamiltonian.

In the present paper we demonstrate the exact solvability of the rational and trigonometric F_4 models for general g, g_1 . The consideration uses a notion and a constructive criterion for exact solvability presented in [4]. This notion is based on the existence of a flag of functional spaces ⁶ with an *explicit* basis preserved by the Hamiltonian. A particular criterion for exact solvability consists of checking whether the flag is related to finite-dimensional representation spaces of a Lie algebra of differential operators. If this criterion is fulfilled, then the Hamiltonian of the given system can be written in terms of the generators of this algebra which is called the *hidden* algebra of the system. In [5] it was shown that the eigenfunctions of the N -body Calogero and Sutherland models [3, 6] form an infinite flag of linear spaces of inhomogeneous polynomials in $(N - 1)$ variables, which coincide to finite-dimensional representation spaces of the algebra $gl(N)$, realized by first order differential operators in symmetric representation. The corresponding Hamiltonians were rewritten as quadratic polynomials in the generators of the maximal affine subalgebra of $gl(N)$, and the coupling constants appear only in the coefficients of these polynomials. Recently, it was shown that this statement can be extended to *all* $ABCD$ Olshanetsky-Perelomov rational and trigonometric integrable systems as well as to their SUSY generalizations which turned out to be associated to the hidden superalgebra $gl(N|N - 1)$ (see Ref. [7]). Later it was shown that for the 3-body Calogero-Sutherland models as well as BC_2 models there exists a specific additional (second) hidden algebra which was called $g^{(2)} \subset \text{diff}(2, \mathbb{R})$. This algebra turned out to be the hidden algebra of the G_2 rational and trigonometric models [8, 9] as well. Thus, the A_2, BC_2, G_2 rational and trigonometric models are characterized by the *same* hidden algebra $g^{(2)}$ and their Hamiltonians can be written as non-linear combination of the $g^{(2)}$ generators. The flags which occurred in all above-mentioned models were always non-classical. Therefore their Hamiltonians were emerged in block-triangular form and the problem of spectra was reduced to a diagonalization of each separate block. However, a remarkable property of all above-mentioned models holds: a certain change of variables preserving the flag was sufficient to diagonalize all blocks simultaneously and, finally, arrive at the pure triangular form.

In the present paper we show that the Hamiltonians of both rational and trigonometric F_4 models admit algebraic form, preserve the same infinite non-classical flag of linear spaces of inhomogeneous polynomials and possess a hidden algebra which we call $f^{(4)} \subset \text{diff}(4, \mathbb{R})$. This algebra is an infinite-dimensional, finitely-generated algebra of differential operators

⁵The form of the Hamiltonian is called *algebraic*, if exists, when the Hamiltonian can be represented as a linear differential operator with polynomial coefficients

⁶A sequence of linear spaces each one embedded into the next one, $\mathcal{P}_1 \subset \mathcal{P}_2 \cdots \subset \mathcal{P}_n \subset \dots$, forms an object called *flag*. A flag is called *infinite flag (filtration)* if a number of these spaces is infinite. A flag is called *classical* if $\dim \mathcal{P}_n = \dim \mathcal{P}_{n-1} + 1$.

possessing a finite-dimensional invariant subspace. It is worth to note that the D_4 rational and trigonometric models, which play an important role in our analysis, are degenerations of BC_4 models from one side and F_4 models from another one. It implies that the D_4 rational and trigonometric models possess *two* different hidden algebras: $gl(5)$ and $f^{(4)}$ as a consequence of these degenerations.

The paper is organized as follows. In Section 2 the $gl(5)$ Lie-algebraic form of the D_4 rational and trigonometric models is studied. In Section 3 the rational F_4 model is analyzed, its algebraic and Lie-algebraic forms are derived. In particular, a quasi-exactly-solvable generalization of the rational F_4 model is found and studied. The trigonometric F_4 model is investigated in section 4. The variables providing the algebraic and the $f^{(4)}$ Lie-algebraic forms of the Hamiltonian are found by averaging over an orbit of the Weyl group. We also discuss an alternative way to found such variables in connection with 'dual' properties of the problem. Transition of the algebraic Hamiltonian from the block-triangular to pure triangular form based on introduction of new variables completes a demonstration of the exact solvability of the model. A realization of the algebra $gl(5)$ in terms of first order differential operators acting on four-dimensional space is given in Appendix A. Appendix B is devoted to a description of the infinite-dimensional algebra of differential operators $f^{(4)}$, admitting finite-dimensional representations in terms of inhomogeneous polynomials in four variables. In Appendix C we give an explicit form of the variables leading to the algebraic form of the F_4 Hamiltonian in z -representation. Finally, in Appendix D the explicit formulas for the first several eigenfunctions of the general F_4 model are presented.

2 Algebraic and Lie-algebraic forms of the D_4 rational and trigonometric models

In this section we represent the Hamiltonians of the D_4 rational and trigonometric models in an algebraic form, by making use of permutationally symmetric coordinates. The Hamiltonians can be written in terms of the generators of the $gl(5)$ algebra [7] and thus they have the $gl(5)$ Lie-algebraic form.

2.1 The rational D_4 model

The Hamiltonian of the D_4 rational model is defined by (see [1])

$$\mathcal{H}_{D_4}^{(r)}(x) = \frac{1}{2} \sum_{i=1}^4 [-\partial_{x_i}^2 + 4\omega^2 x_i^2] + 2g \sum_{i<j}^4 \left[\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right] \quad (2.1)$$

where $g = \nu(\nu - 1)/2 > -1/8$ is the coupling constant and ω is the harmonic oscillator frequency. The ground state eigenfunction of the Hamiltonian (2.1) is given by

$$\Psi_0^{(r)} \equiv \exp(-\Phi_0^{(r)}) = (\Delta_- \Delta_+)^{\nu} \exp\left(-\omega \sum_{i=1}^4 x_i^2\right), \quad (2.2)$$

with

$$\Delta_{\pm} = \prod_{j < i}^4 (x_i \pm x_j), \quad (2.3)$$

and the ground state energy is

$$E_0 = 4\omega(1 + 6\nu). \quad (2.4)$$

As a first step towards an algebraic form of (2.1) let us make a gauge rotation

$$h_{D_4}^{(r)} = -2(\Psi_0^{(r)}(x))^{-1}(\mathcal{H}_{D_4}^{(r)} - E_0)\Psi_0^{(r)}(x). \quad (2.5)$$

Then, in order to code permutation symmetry $x_i \leftrightarrow x_j$ and reflection symmetry $x_i \rightarrow -x_i$ of the Hamiltonian ⁷, we take as new coordinates the elementary symmetric (Vieta) polynomials S_i of the arguments x_i^2 ,

$$\begin{aligned} s_1 &= S_1(x^2) = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ s_2 &= S_2(x^2) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_3^2 + x_2^2 x_4^2 + x_3^2 x_4^2, \\ s_3 &= S_3(x^2) = x_1^2 x_2^2 x_3^2 + x_1^2 x_2^2 x_4^2 + x_1^2 x_3^2 x_4^2 + x_2^2 x_3^2 x_4^2, \\ s_4 &= S_4(x^2) = x_1^2 x_2^2 x_3^2 x_4^2. \end{aligned} \quad (2.6)$$

In the s variables the Hamiltonian $h_{D_4}^{(r)}$ becomes

$$h_{D_4}^{(r)} = \sum_{i,j=1}^4 A_{ij} \frac{\partial}{\partial s_i} \frac{\partial}{\partial s_j} + \sum_{j=1}^4 (B_j + C_j(\omega, \nu)) \frac{\partial}{\partial s_j}, \quad (2.7)$$

where

$$A_{ij} = \sum_{k=1}^4 \frac{\partial s_i}{\partial x_k} \frac{\partial s_j}{\partial x_k} = 4 \sum_{l \geq 0} (2l + 1 + j - i) s_{i-l-1} s_{j+l},$$

⁷These symmetries correspond to the group of automorphisms of the root space which in the D_4 case is broader than the Weyl group.

$$A_{ji} = A_{ij} ,$$

$$\begin{aligned} B_j + C_j(\omega, \nu) &= \sum_{k=1}^4 \left(\frac{\partial^2 s_i}{\partial x_k^2} \right) - \frac{1}{2} \sum_{k=1}^4 \frac{\partial \Phi_0^{(r)}}{\partial x_k} \frac{\partial s_j}{\partial x_k} \\ &= 2[1 + 2\nu(4 - j)](5 - j)s_{j-1} - 8\omega j s_j , \end{aligned} \quad (2.8)$$

where $s_0 = 1$ and $s_i = 0$ at $i > 4$ or $i < 0$.

Since the coefficients (2.8) are polynomials in s , the operator (2.7) gives the *algebraic* form of the D_4 rational Hamiltonian. It is worth to note that the coefficient matrix A_{ij} makes sense of a flat space metric. An important feature of (2.8) is that the coefficients A_{ij} and B_j are the second and the first order polynomials in s coordinates, respectively. It can be shown that it leads to two important conclusions: (i) the operator (2.7) preserves the flag of spaces of polynomials

$$\mathcal{P}_n^{(D_4)} = \langle s_1^{p_1} s_2^{p_2} s_3^{p_3} s_4^{p_4} | 0 \leq p_1 + p_2 + p_3 + p_4 \leq n \rangle , \quad (2.9)$$

with the characteristic vector ⁸

$$\vec{f} = (1, 1, 1, 1) , \quad (2.10)$$

and hence the operator (2.7) possesses infinitely many finite-dimensional invariant subspaces, and (ii) the operator (2.7) has the $gl(5)$ Lie-algebraic form [7], since it can be rewritten in terms of the $gl(5)$ algebra generators but without raising generators J_i^+ (see Appendix A). If $\nu = 0$, $\omega = 0$, the operator $h_{D_4}^{(r)}$ becomes the flat space Laplacian written in the $gl(5)$ Lie-algebraic form.

The operator (2.7) with coefficients (2.8) appears to be in pure triangular form with respect to the action on basis of monomials $s_1^{p_1} s_2^{p_2} s_3^{p_3} s_4^{p_4}$. Therefore, the spectrum of (2.7), $h_{D_4}^{(r)} \varphi = -2\epsilon \varphi$, can be found explicitly and is equal to

$$\epsilon_n = 4\omega(p_1 + 2p_2 + 3p_3 + 4p_4) , \quad (2.11)$$

where $n = 0, 1, \dots$ and p_i are non-negative integers with a condition $p_1 + p_2 + p_3 + p_4 = n$. The spectrum does not depend on the coupling constant g , it is equidistant and corresponds to the spectrum of the harmonic oscillator. Degeneracy of the spectrum is related to the number of partitions of an integer number n to $p_1 + p_2 + p_3 + p_4$. The spectrum of the original rational D_4 Hamiltonian (2.1) is $E_n = E_0 + \epsilon_n$.

⁸A term ‘characteristic vector’ was proposed in [10]. It is a vector with components which are equal to the coefficients in front of p_i .

It is worth to mention that the boundaries of configuration space are determined by zeros of the ground state wave function (2.2). In s -variables the boundary is an algebraic surface in four variables

$$\begin{aligned}
(\Delta_+ \Delta_-)^2 &= \prod_{i < j} (x_i^2 - x_j^2)^2 = 256s_4^3 - s_4^2(192s_3s_1 + 128s_2^2 - 144s_2s_1^2 + 27s_1^4) \\
&\quad + 2s_4(72s_3^2s_2 - 3s_3^2s_1^2 - 40s_3s_2^2s_1 + 9s_3s_2s_1^3 + 8s_2^4 - 2s_2^3s_1^2) \\
&\quad - 27s_3^4 + 2s_1s_3^3(9s_2 - 2s_1^2) - s_2^2s_3^2(4s_2 - s_1^2) = 0 .
\end{aligned} \tag{2.12}$$

A simple relation between Jacobian and pre-exponential factor in the ground state wave function (2.12) exists

$$\left[\det \left(\frac{\partial s_i}{\partial x_k} \right) \right]^2 = 256 (\Delta_+ \Delta_-)^2 s_4 . \tag{2.13}$$

Such a simplicity is of a general character because the Jacobian for the transformation from x 's to the basic Weyl-invariant polynomials is equal (up to constant factor) to the product of linear functions vanishing on hyperplanes corresponding to roots (see, for instance, [11]), i.e. just to $\Delta_+ \Delta_-$ from (2.3). Since our s -variables (2.6) differ from the D_4 basic invariants by taking $s_4 = (x_1 x_2 x_3 x_4)^2$ instead of $x_1 x_2 x_3 x_4$, the extra factor s_4 appears in the squared Jacobian.

2.2 The trigonometric D_4 model

The Hamiltonian of the D_4 trigonometric model has the form [1]

$$\mathcal{H}_{D_4}^{(t)} = -\frac{1}{2} \sum_{i=1}^4 \frac{\partial^2}{\partial x_i^2} + 2g\beta^2 \sum_{i < j}^4 \left[\frac{1}{\sin^2(\beta(x_i - x_j))} + \frac{1}{\sin^2(\beta(x_i + x_j))} \right] \tag{2.14}$$

where $g = \nu(\nu - 1)/2 > -1/8$ and β is a parameter. When β tends to zero the Hamiltonian (2.14) coincides with (2.1) at $\omega = 0$.

The ground state wave function is [2, 12]

$$\Psi_0^{(t)} = (\Delta_-(x, \beta) \Delta_+(x, \beta))^\nu = \beta^{-12\nu} \prod_{i < j} |\sin^2 \beta x_i - \sin^2 \beta x_j|^\nu , \tag{2.15}$$

with

$$\Delta_\pm(x, \beta) = \beta^{-6} \prod_{i < j} \sin \beta(x_i \pm x_j) , \tag{2.16}$$

and the ground state energy equals

$$E_0 = 28\beta^2\nu^2 . \quad (2.17)$$

Using the same approach as in Section 2.1, we make a gauge rotation of (2.14) with the ground state eigenfunction as a gauge factor, $h_{D_4}^{(t)} = -2\Psi_0^{-1}(\mathcal{H}_{D_4}^{(t)} - E_0)\Psi_0$. A straightforward calculation leads to the operator

$$h_{D_4}^{(t)} = \sum_{i=1}^4 \partial_i^2 + \nu \sum_{i<j} [\cot(\beta(x_i - x_j))(\partial_i - \partial_j) + \cot(\beta(x_i + x_j))(\partial_i + \partial_j)] \quad (2.18)$$

In order to code the permutation symmetry $x_i \leftrightarrow x_j$ and reflection symmetry $x_i \rightarrow -x_i$, as well as the periodicity of the Hamiltonian, we introduce new coordinates as the elementary symmetric polynomials of the trigonometric arguments

$$\sigma_k(x) = S_k(y_i^2) , \quad (2.19)$$

$$y_i = \frac{\sin \beta x_i}{\beta} , \quad (2.20)$$

which in the limit $\beta \rightarrow 0$ coincide with (2.6). In these variables the Hamiltonian $h_{D_4}^{(t)}$ becomes

$$h_{D_4}^{(t)} = \sum_{i,j=1}^4 A_{ij} \frac{\partial}{\partial \sigma_i} \frac{\partial}{\partial \sigma_j} + \sum_{j=1}^4 (B_j + C_j(\nu)) \frac{\partial}{\partial \sigma_j} \quad (2.21)$$

where

$$\begin{aligned} A_{ij} &= 4 \sum_{l \geq 0} (2l + 1 + j - i) \sigma_{i-l-1} \sigma_{j+l} \\ &\quad - 4\beta^2 [i\sigma_i \sigma_j + (i - j - 2)\sigma_{i-1} \sigma_{j+1} + (i - j - 4)\sigma_{i-2} \sigma_{j+2}] \quad \text{for } i \leq j , \\ A_{ji} &= A_{ij} , \\ B_j + C_j(\nu) &= 2[1 + 2\nu(4 - j)](5 - j)\sigma_{j-1} - 4\beta^2 j[\sigma_j + (7 - j)\nu\sigma_{j+1}] . \end{aligned} \quad (2.22)$$

It is assumed that $\sigma_0 = 1$, and $\sigma_i = 0$ at $i < 0$ or $i > 4$. The operator (2.21) with coefficients (2.22) gives the *algebraic* form of the D_4 trigonometric Hamiltonian. It is worth to emphasize that A_{ij} makes sense of a one-parametric flat space metric, which is reduced to (2.8) at $\beta = 0$. Similarly to what appeared in previously discussed D_4 rational case the coefficients A_{ij} , B_j are polynomials in σ 's of second and first degree, respectively.

Hence, the Hamiltonian $h_{D_4}^{(t)}$ can be written in terms of $gl(5)$ generators realized as first order differential operators [4] (see Appendix A), producing the $gl(5)$ *Lie-algebraic* form of the D_4 trigonometric Hamiltonian. Furthermore, it can be easily verified that the $gl(5)$ Lie-algebraic form of (2.14) does not contain the raising generators J_i^+ . Hence the operator (2.21) with coefficients (2.22) preserves the same flag $\mathcal{P}^{(D_4)}$ (2.9) of the spaces of polynomials but in σ variables with the same characteristic vector (2.10). If $\nu = 0$, the operator $h_{D_4}^{(t)}$ becomes the flat space Laplacian written in $gl(5)$ Lie-algebraic form depending on a single free parameter β ⁹.

The operator (2.21) with coefficients (2.22) appears to be in pure triangular form with respect to the action on basis of monomials $\sigma_1^{p_1} \sigma_2^{p_2} \sigma_3^{p_3} \sigma_4^{p_4}$. Therefore the spectrum of (2.21), $h_{D_4}^{(t)} \varphi = -2\epsilon \varphi$, can be found explicitly

$$\begin{aligned} \epsilon_n = & 2\beta^2 [5p_1(p_1 + 2p_2 + 2p_3 + 2p_4) + 10p_2(p_2 + 2p_3 + 2p_4) + 15p_3(p_3 + 2p_4) \\ & + 20p_4^2 - 3(p_1 + 2p_2 + 3p_3 + 4p_4) + 4\nu(3p_1 + 5p_2 + 6p_3 + 6p_4)] , \end{aligned} \quad (2.23)$$

where $n = 0, 1, \dots$, and p_i are non-negative integers with a condition $(p_1 + p_2 + p_3 + p_4) = n$. The spectrum of the original trigonometric D_4 Hamiltonian (2.14) is $E_n = E_0 + \epsilon_n$. This completes a demonstration of the exact solvability of the trigonometric D_4 model.

The boundaries of configuration space are determined by zeros of the ground state wave function (2.15). In the σ variables the boundary is an algebraic surface in four-dimensional space

$$\begin{aligned} (\Delta_-(x, \beta) \Delta_+(x, \beta))^2 &= \beta^{-24} \prod_{i < j} (\sin^2 \beta x_i - \sin^2 \beta x_j)^2 \\ &= 16 \sigma_4 \sigma_2^4 - 4 (\sigma_1^2 \sigma_4 + \sigma_3^2) \sigma_2^3 - (80 \sigma_4 \sigma_1 \sigma_3 - \sigma_1^2 \sigma_3^2 + 128 \sigma_4^2) \sigma_2^2 \\ &\quad + 18 (\sigma_1^3 \sigma_3 \sigma_4 + 8 \sigma_4 \sigma_3^2 + 8 \sigma_4^2 \sigma_1^2 + \sigma_3^3 \sigma_1) \sigma_2 - 27 \sigma_3^4 - 6 \sigma_3^2 \sigma_1^2 \sigma_4 \\ &\quad - 27 \sigma_1^4 \sigma_4^2 - 4 \sigma_1^3 \sigma_3^3 - 192 \sigma_4^2 \sigma_1 \sigma_3 + 256 \sigma_4^3 = 0 \end{aligned} \quad (2.24)$$

(cf. (2.12)). We should emphasize that the algebraic surface (2.24) does not depend on the parameter β and therefore equation (2.24) defines the *same* surface as in equation (2.12) but in the space of the σ -variables. It means that the configuration space of both rational and trigonometric D_4 problems is the *same* when is written in appropriate variables.

There exists a trigonometric generalization of (2.13) giving a connection between Ja-

⁹The operators (2.7) with coefficients (2.8) and (2.21) with coefficients (2.22) can be considered as two different deformations of Laplacian which preserve the same flag (2.9). An interesting question is about existence of other non-trivial deformations of Laplacian preserving the same flag.

cobian and (2.24)

$$\left[\det \left(\frac{\partial \sigma_i}{\partial x_k} \right) \right]^2 = 256 (\Delta_+(\beta) \Delta_-(\beta))^2 \sigma_4 (1 - \beta^2 \sigma_1 + \beta^4 \sigma_2 - \beta^6 \sigma_3 + \beta^8 \sigma_4) . \quad (2.25)$$

In the limit $\beta \rightarrow 0$ the equation (2.25) becomes (2.13).

3 The rational F_4 model

We shall derive in this section the algebraic and Lie-algebraic forms of the rational F_4 models which lead to polynomial eigenfunctions. This fact together with the explicit calculation of the eigenvalues, exhibits the exact solvability of the model.

3.1 Algebraic form

The ground state of the rational F_4 model

$$\begin{aligned} \mathcal{H}_{F_4}^{(r)} = & \frac{1}{2} \sum_{i=1}^4 (-\partial_{x_i}^2 + 4\omega^2 x_i^2) + 2g \sum_{j>i} \left(\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right) \\ & + 2g_1 \sum_{i=1}^4 \frac{1}{x_i^2} + 8g_1 \sum_{\nu's=0,1} \frac{1}{[x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4]^2} , \end{aligned} \quad (3.1)$$

can be written as

$$\Psi_0^{(r)}(x) = (\Delta_- \Delta_+)^{\nu} (\Delta_0 \Delta)^{\mu} \exp \left(-\omega \sum_{i=1}^4 x_i^2 \right) , \quad (3.2)$$

where $g = \nu(\nu - 1)/2$, $g_1 = \mu(\mu - 1)$, and

$$\begin{aligned} \Delta_{\pm} &= \prod_{j<i}^4 (x_i \pm x_j) , \\ \Delta_0 &= 2^4 \prod_{i=1}^4 x_i , \\ \Delta &= \prod_{\nu's=0,1} [x_1 + (-1)^{-\nu_2} x_2 + (-1)^{-\nu_3} x_3 + (-1)^{-\nu_4} x_4] , \end{aligned} \quad (3.3)$$

while the ground state energy is

$$E_0 = 4\omega(1 + 6\mu + 6\nu) . \quad (3.4)$$

The transformation (1.8) demonstrates the ‘dual relation’ between two parts of the wave function

$$\begin{aligned}(\Delta_+ \Delta_-)^2(x) &= (\Delta_0 \Delta)^2(z) , \\ (\Delta_0 \Delta)^2(x) &= (\Delta_+ \Delta_-)^2(z) .\end{aligned}\tag{3.5}$$

General statement of the Hamiltonian reduction method is that any eigenfunction of (3.1) can be written in a factorizable form as

$$\Psi(x) = \Psi_0^{(r)}(x) P_{F_4}(x) ,\tag{3.6}$$

where $P_{F_4}(x)$ is a polynomial in x_i ’s. The operator having these polynomials as eigenfunctions can be obtained by gauge rotation of (3.1):

$$h_{F_4}^{(r)} = -2(\Psi_0^{(r)}(x))^{-1}(\mathcal{H}_{F_4}^{(r)} - E_0)\Psi_0^{(r)}(x) .\tag{3.7}$$

In order to find variables supposedly giving an algebraic form to the original F_4 Hamiltonian (3.1) we consider as a criterion of their choice the invariance with respect to the symmetries of the Hamiltonian [5, 10], i.e. under the group of automorphisms A of the F_4 root space (in the F_4 case this group coincides with the Weyl group W). The invariant polynomials of the *lowest* possible degrees generate the algebra S^W of W -invariant polynomials. We denote these polynomials $t_a(x)$. They can be found by averaging over some group orbit elementary polynomials $(\alpha, x)^{2a}$ (we used the orbit Ω generated by the root $e_1 + e_2$, other orbits give algebraically related invariants):

$$t_a(x) = \sum_{\alpha \in \Omega} (\alpha, x)^{2a} , \quad a = 1, 3, 4, 6 .\tag{3.8}$$

Their powers $2a = 2, 6, 8, 12$ are the *degrees* of the group W .

The basis of t_a ’s allows non-linear transformations which preserve the Weyl invariance

$$\begin{aligned}t_3 &\rightarrow t_3 + a_3 t_1^3 , \\ t_4 &\rightarrow t_4 + a_4 t_1^4 + b_4 t_1 t_3 , \\ t_6 &\rightarrow t_6 + a_6 t_1^6 + b_6 t_1^3 t_3 + c_6 t_1^2 t_4 + d_6 t_3^2 ,\end{aligned}\tag{3.9}$$

where $a_{3,4,6}$, $b_{4,6}$, c_6 , d_6 are any numbers. Supposedly, these transformations do not destroy an algebraic form of the operator $h_{F_4}^{(r)}$ if it appears in variables (3.8). These transformations are nothing but automorphisms of the algebra S^W .

Finally, we choose the following variables

$$t_1 = s_1 ,$$

$$\begin{aligned}
t_3 &= s_3 - \frac{1}{6} s_1 s_2 \\
t_4 &= s_4 - \frac{1}{4} s_1 s_3 + \frac{1}{12} s_2^2 \\
t_6 &= s_4 s_2 - \frac{1}{36} s_2^3 - \frac{3}{8} s_3^2 + \frac{1}{8} s_1 s_2 s_3 - \frac{3}{8} s_1^2 s_4 .
\end{aligned} \tag{3.10}$$

where s 's are defined in (2.6)). The Hamiltonian (3.7) written in the coordinates (3.10) has an algebraic form confirming our expectation

$$h_{\text{F}_4}^{(r)} = \sum_{a,b} A_{ab} \frac{\partial^2}{\partial t_a \partial t_b} + \sum_a (B_a + C_a(\mu, \nu) + C_a(\omega)) \frac{\partial}{\partial t_a} , \tag{3.11}$$

where summation goes over $a, b = 1, 3, 4, 6$ and the coefficient functions are

$$\begin{aligned}
A_{11} &= 4 t_1 & A_{13} &= 12 t_3 , \\
A_{14} &= 16 t_4 , & A_{16} &= 24 t_6 , \\
A_{33} &= -\frac{2}{3} t_1^2 t_3 + \frac{20}{3} t_1 t_4 , & A_{34} &= -\frac{4}{3} t_1^2 t_4 + 8 t_6 , \\
A_{36} &= 16 t_4^2 - 2 t_1^2 t_6 , & A_{44} &= -4 t_3 t_4 - 2 t_1 t_6 , \\
A_{46} &= -4 t_1 t_4^2 - 6 t_3 t_6 , & A_{66} &= -12 t_3 t_4^2 - 6 t_1 t_4 t_6 , \\
A_{ba} &= A_{ab} ,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
B_1 &= 8 , & B_3 &= -t_1^2 , \\
B_4 &= -4 t_3 , & B_6 &= -8 t_1 t_4 .
\end{aligned} \tag{3.13}$$

The coefficients A_{ab} have a meaning of elements of metric with upper indices which corresponds to the flat space. Thus, the operator (3.11) with the coefficients A_{ab} and B_a only ($C_a(\mu, \nu) = C_a(\omega) = 0$) is the flat space Laplace operator.

Terms stemming from the potential part of the Hamiltonian are proportional to μ, ν ,

$$\begin{aligned}
C_1(\mu, \nu) &= 48 (\nu + \mu) , & C_3(\mu, \nu) &= -2 (2\nu + \mu) t_1^2 , \\
C_4(\mu, \nu) &= -12 \nu t_3 , & C_6(\mu, \nu) &= -12 \nu t_1 t_4 ,
\end{aligned} \tag{3.14}$$

(cf. (3.13)) and also to ω

$$C_1(\omega) = -4 \omega t_1 , \quad C_3(\omega) = -12 \omega t_3 ,$$

$$C_4(\omega) = -16\omega t_4, \quad C_6(\omega) = -24\omega t_6. \quad (3.15)$$

The operator (3.11) with the coefficients A_{ab} , B_a , $C_a(\mu, \nu)$, and $C_a(\omega)$ represents the *algebraic* form of the F_4 -rational model. It is easy to check that in the t -coordinates the Hamiltonian $h_{F_4}^{(r)}$ preserves a flag of polynomials $\mathcal{P}^{(F_4)}$, given by

$$\mathcal{P}_n^{(F_4)} = \langle t_1^{p_1} t_3^{p_3} t_4^{p_4} t_6^{p_6} \mid 0 \leq p_1 + 2p_3 + 2p_4 + 3p_6 \leq n \rangle, \quad (3.16)$$

with the characteristic vector \vec{f}

$$\vec{f} = (1, 2, 2, 3), \quad (3.17)$$

(cf. (2.10)). This vector reminds the highest root among short ones in the root system of the algebra F_4 written in the basis of simple roots¹⁰. The flag (3.16) remains invariant under non-linear transformations (3.9). The Hamiltonian also continues to be algebraic under these transformations. We should mention that the first study of the algebraic form of F_4 model was carried out in [10] using a set of variables with the only difference from our variables (3.10) in t_6 ¹¹. It was found the characteristic vector $(1, 2, 3, 5)$ in variance to (3.17). However, since different t_6 variable is of the form (3.9), it preserves the same flag $\mathcal{P}^{(F_4)}$ (3.16). It implies that the flag indicated in [10] is not minimal in variance to the statement made in this article.

We were able to find one-parametric algebra of differential operators for which there exists finite-dimensional irreducible representation marked by an integer value of the parameter. Furthermore, the finite-dimensional representation spaces corresponding to different integer values of this parameter form the infinite non-classical flag $\mathcal{P}^{(F_4)}$ (3.16) (see Appendix B). We call this algebra $f^{(4)}$. Like the algebra $g^{(2)}$ (see [8]), the algebra $f^{(4)}$ is infinite-dimensional but finitely-generated. The F_4 rational Hamiltonian in the algebraic form (3.11) can be rewritten in terms of the generators of this algebra.

The operator (3.11) with coefficients (3.12)-(3.15) appears to be in pure triangular form with respect to the action on monomials $\sigma_1^{p_1} \sigma_2^{p_2} \sigma_3^{p_3} \sigma_4^{p_4}$. One can find the spectrum of (3.11), $h_{F_4}^{(r)} \varphi = -2\epsilon \varphi$, explicitly (cf. (2.11))

$$\epsilon_n = 2\omega(p_1 + 3p_2 + 4p_3 + 6p_4), \quad (3.18)$$

where $n = 0, 1, \dots$, and p_i are non-negative integers with a condition $p_1 + 2p_2 + 2p_3 + 3p_4 = n$. The spectrum does not depend on the coupling constants g , g_1 , is equidistant and corresponds (with different degeneracy) to the spectrum of the harmonic oscillator as well

¹⁰We thank Victor Kač for this remark. The similar statement holds for A_n -Calogero-Sutherland models where the flags are characterized by $\vec{f} = (1, 1, \dots, 1)$ [5] and G_2 models where $\vec{f} = (1, 2)$ [8].

¹¹It corresponds to $a_6 = 0, c_6 = 3/8, b_6 = 3/32, d_6 = 3/8$.

as the rational D_4 model. Finally, the energies of the original rational F_4 Hamiltonian (3.1) are $E_n = E_0 + \epsilon_n$.

Configuration space of the rational F_4 model (3.1) is defined by zeros of the ground state eigenfunction, i.e. by zeros of the pre-exponential factor in (3.2). These zeros also define boundaries of Weyl chamber (see [2]). The squared pre-exponential factor can be written as a product of two factors. The first one

$$(\Delta_+ \Delta_-)^2 = -192 t_6^2 + 256 t_4^3, \quad (3.19)$$

corresponds to the rational D_4 model (2.1) appearing at $g_1 = 0$ (cf. (2.12)). It looks much simpler than in [10]. The second factor

$$\begin{aligned} (\Delta_0 \Delta)^2 = & -3072 t_6^2 + 4096 t_4^3 - 2304 t_3^2 t_6 - 432 t_3^4 + 3072 t_1 t_3 t_4^2 \\ & - 768 t_1^2 t_4 t_6 + 480 t_1^2 t_3^2 t_4 - 192 t_1^3 t_3 t_6 - 8 t_1^3 t_3^3 \\ & + 16 t_1^4 t_4^2 + 8 t_1^5 t_3 t_4 - \frac{8}{3} t_1^6 t_6, \end{aligned} \quad (3.20)$$

corresponds to a case of the degenerate F_4 model, $g = 0$ (as it was noted in Introduction it is equivalent to the D_4 model in dual variables). Thus, a boundary of the configuration space of the rational F_4 model is confined by the algebraic surfaces (3.19)–(3.20) of the third and seventh orders, correspondingly¹², being in total given by the algebraic surface of the tenth order.

According to general theory (see [11]) the relation between Jacobian and the pre-exponential factor in the ground-state wave function (3.2) is straightforward:

$$\left[\det \left(\frac{\partial t_a}{\partial x_k} \right) \right]^2 = \frac{1}{4096} (\Delta_+ \Delta_-)^2 (\Delta_0 \Delta)^2. \quad (3.21)$$

In order to find eigenfunctions one can derive recurrence relations and then solve them out. For several first eigenfunctions it can be done explicitly (see examples in Appendix D). Similar to what was found previously for both the Calogero (A_n -rational) model [3, 5] and the G_2 rational model [13, 9] there exists among eigenfunctions a family $\Phi_n(t_1)$ which depends on a single variable t_1 . This fact was already used in [14] in order to construct quasi-exactly-solvable many-body generalizations of the Calogero model (for definition of quasi-exact-solvability, see for example [15]). This family of eigenstates appears due to a fact that the coefficients A_{11} , B_1 , C_1 in (3.12) – (3.15) depend on the single variable t_1

¹²In coordinates proposed in [10] these surfaces appear as of the eighth and tenth orders, correspondingly, leading to the total algebraic surface of the 18th order. We classify this choice of variables as *non-minimal*.

only. Therefore, it is easy to verify that beside the flag $\mathcal{P}^{(F_4)}$ (3.16) the Hamiltonian $h_{F_4}^{(r)}$ preserves another flag of polynomials $\mathcal{P}^{(1)}$, defined by the spaces

$$\mathcal{P}_n^{(1)}(t_1) = \langle t_1^{p_1} | 0 \leq p_1 \leq n \rangle .$$

Since $\mathcal{P}_n^{(1)} \subset \mathcal{P}_n^{(F_4)}$ for any n , then the flag $\mathcal{P}^{(1)} \subset \mathcal{P}^{(F_4)}$. It leads to a degeneration of the general spectral solution for the operator $h_{F_4}^{(r)}$ to an equation

$$4t_1\Phi'' - [4\omega t_1 - 8(6\mu + 6\nu + 1)]\Phi' = E\Phi , \quad (3.22)$$

which can be solved explicitly,

$$\Phi_n^{(r)} = L_n^{(12\mu+12\nu+1)}(\omega t_1) , \quad E_n = -4\omega n , \quad (3.23)$$

where $L_n^{(a)}$ is a Laguerre polynomial in a standard notation, $n = 0, 1, 2, \dots$. Existence of the flag $\mathcal{P}^{(1)}$ (which looks like as truly minimal flag) is a consequence of very degenerate nature of the F_4 rational model. The Hamiltonian of the F_4 trigonometric model does not preserve this flag. Actually, there exist other ‘degenerate’ flags preserved by the F_4 rational Hamiltonian and thus other infinite families of eigenstates depending on (t_1, t_3) or (t_1, t_3, t_4) variables only, which can be found explicitly. They are also a consequence of degenerate nature of the F_4 rational model and a reader can easily investigate them.

3.2 Quasi-exactly-solvable generalization of the F_4 rational model

Above-mentioned remarkable property of $h_{F_4}^{(r)}$ allowing a family of eigenfunctions depending on one variable leads to a possibility to construct a quasi-exactly-solvable (QES) generalization (for discussion see, for example, [15]) of the rational F_4 model. In order to do this we will use the same trick as was used in [14]. We look for a QES generalization of (1.1) of a form

$$\mathcal{H}_{F_4}^{(qes)} = \mathcal{H}_{F_4}^{(r)} + V^{(qes)}(t_1) . \quad (3.24)$$

Let us make a gauge rotation (3.24) in the form (3.7) and then require that the resulting operator possesses t_1 -depending family of eigenfunctions. It results in the equation

$$h^{(qes)}\Phi \equiv \{4t_1\partial_{t_1}^2 - [4\omega t_1 - 8(6\mu + 6\nu + 1)]\partial_1 - 2V^{(qes)}\}\Phi = -2\epsilon\Phi . \quad (3.25)$$

where the spectral parameter ϵ is related to energy of the Hamiltonian (3.24) through $E_n = E_0 + \epsilon_n$ with E_0 given by (3.4).

Now one can pose a question under what condition on potential $V^{(qes)}$, the operator $h^{(qes)}$ is Lie-algebraic. The problem is similar to which appeared in [14] and the solution

is the following: make a gauge rotation of $h^{(qes)}$ in such a way that (i) to get rid off the potential $V^{(qes)}$, and (ii) to obtain the $sl(2)$ Lie-algebraic form

$$\begin{aligned} h_{sl(2)}^{(qes)}(t_1) &= t_1^{-\gamma} \exp\left(\frac{a}{4}t_1^2\right) h^{(qes)} t_1^{\gamma} \exp\left(-\frac{a}{4}t_1^2\right) \\ &= 4J_n^0 J^- - 4\omega J_n^0 + 2[n + 4(6\mu + 6\nu + 1)]J^- + 4aJ_n^+ - 4\gamma J^- . \end{aligned} \quad (3.26)$$

Here

$$J_n^+ = \tau_1^2 \partial_1 - nt_1 , \quad J_n^0 = t_1 \partial_1 - \frac{n}{2} , \quad J^- = \partial_1 , \quad (3.27)$$

are the generators of the $sl(2)$ algebra, and the potential $V^{(qes)}$ should be

$$\begin{aligned} V^{(qes)} &= a^2 t_1^3 - 2a\omega t_1^2 + 2a[2n + 1 - \gamma + 2(6\mu + 6\nu + 1)]t_1 \\ &\quad + \frac{2\gamma[\gamma + 1 - 2(6\mu + 6\nu + 1)]}{t_1} , \end{aligned} \quad (3.28)$$

where the constant terms in potential are dropped off. Finally, we arrive in x -variables at the Hamiltonian ($\mathbf{x}^2 = \sum_{i=1}^4 x_i^2$)

$$\begin{aligned} \mathcal{H}_{F_4}^{(qes)} &= \frac{1}{2} \sum_{i=1}^4 (-\partial_{x_i}^2 + 4\omega^2 x_i^2) + 2g \sum_{j>i} \left(\frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} \right) \\ &\quad + 2g_1 \sum_{i=1}^4 \frac{1}{x_i^2} + 8g_1 \sum_{\nu, s=0,1} \frac{1}{[x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4]^2} \\ &\quad + a^2 (\mathbf{x}^2)^3 - 2a\omega (\mathbf{x}^2)^2 + 2a[2n + 1 - \gamma + 2(6\mu + 6\nu + 1)]\mathbf{x}^2 \\ &\quad + \frac{2\gamma[\gamma + 1 - 2(6\mu + 6\nu + 1)]}{\mathbf{x}^2} , \end{aligned} \quad (3.29)$$

where we know $(n + 1)$ eigenstates explicitly. Their eigenfunctions are of the form

$$\Psi_0^{(r)}(x) = (\Delta_- \Delta_+)^{\nu} (\Delta_0 \Delta)^{\mu} (\mathbf{x}^2)^{\gamma} P_n(\mathbf{x}^2) \exp \left[-\omega \mathbf{x}^2 - \frac{a}{4} (\mathbf{x}^2)^2 \right] , \quad (3.30)$$

where P_n is a polynomial of degree n . Hence we constructed the $sl(2)$ QES deformation of the F_4 rational model. If in (3.29) the parameter g_1 (and, hence, μ) vanishes, we arrive at the $sl(2)$ QES generalization of the D_4 rational model. The latter differs from the $sl(5)$ QES deformation found in [16].

To conclude a discussion of the rational case, one can state that the rational F_4 model admits the algebraic and also the $f^{(4)}$ Lie-algebraic forms. Since the Lie-algebraic form of (3.11) with coefficients (3.12)-(3.15) contain no positive-grading generators, an infinite family of ν, μ -depending polynomial eigenfunctions of (3.11) occurs.

4 The trigonometric F_4 model

4.1 Algebraic form

Let us consider the trigonometric F_4 system now. Its Hamiltonian can be represented as

$$\mathcal{H}_{F_4}^{(t)}(x) = -\frac{1}{2} \sum_{i=1}^4 \partial_{x_i}^2 + 2gV_1(x, \beta) + \frac{g_1}{2} V_2(x, 2\beta) , \quad (4.1)$$

where $g = \nu(\nu - 1)/2$, $g_1 = \mu(\mu - 1)$, and

$$V_1(x, \beta) = \beta^2 \sum_{j>i} \left(\frac{1}{\sin^2 \beta(x_i - x_j)} + \frac{1}{\sin^2 \beta(x_i + x_j)} \right) , \quad (4.2)$$

$$\begin{aligned} V_2(x, 2\beta) &= 4\beta^2 \sum_{i=1}^4 \frac{1}{\sin^2 2\beta x_i} \\ &+ 4\beta^2 \sum_{\nu's=0,1}^4 \frac{1}{\sin^2 \beta [x_1 + (-1)^{\nu_2} x_2 + (-1)^{\nu_3} x_3 + (-1)^{\nu_4} x_4]} . \end{aligned} \quad (4.3)$$

The ground state of the trigonometric F_4 model (4.1)–(4.3) has the form

$$\Psi_0^{(t)}(x, \beta) = (\Delta_+(x, \beta) \Delta_-(x, \beta))^\nu (\Delta_0(x, 2\beta) \Delta(x, 2\beta))^\mu , \quad (4.4)$$

where

$$\begin{aligned} \Delta_\pm(x, \beta) &= \beta^{-6} \prod_{j<i} \sin \beta(x_i \pm x_j) , \\ \Delta_0(x, 2\beta) &= \beta^{-4} \prod_i \sin 2\beta x_i , \\ \Delta(x, 2\beta) &= \beta^{-8} \prod_{\nu's=0,1} \sin \beta [x_1 + (-1)^{-\nu_2} x_2 + (-1)^{-\nu_3} x_3 + (-1)^{-\nu_4} x_4] . \end{aligned} \quad (4.5)$$

Here $\Delta_\pm(x, \beta)$, $\Delta_0(x, 2\beta)$, $\Delta(x, 2\beta)$ are the trigonometric analogs of the factors appearing in the rational case (see (3.2)). In the limit $\beta \rightarrow 0$ they coincide with those of the rational case, $\Delta_\pm(x, 0) = \Delta_\pm(x)$, $\Delta_0(x, 0) = \Delta_0(x)$, $\Delta(x, 0) = \Delta(x)$. The ground state energy of the Hamiltonian (4.1) is given by (cf. (2.15))

$$E_0 = 4\beta^2(7\nu^2 + 14\mu^2 + 18\nu\mu) . \quad (4.6)$$

Guided by general theory [1, 2] and experience gained with previous studies of the Calogero-Sutherland models [5, 7], G_2 model [8] and the rational F_4 model (see Section

3), let us check first whether there exists a family of factorized eigenfunctions of (4.1) of the type $\Psi(x) = \Psi_0^{(t)}(x)P_{F_4}(x)$, where the P_{F_4} are polynomials in some variables. If this is the case, there is a chance that, in accordance with the conjectures made in [4], the trigonometric F_4 model also possesses a hidden algebraic structure. In order to verify this we make first the gauge transformation of (4.1) with the ground state eigenfunction (4.4) as the gauge factor,

$$h_{F_4}^{(t)} = -2(\Psi_0^{(t)}(x))^{-1}(\mathcal{H}_{F_4}^{(t)} - E_0)(\Psi_0^{(t)}(x)) . \quad (4.7)$$

Crucially important step is to find variables (if exist) which lead to an algebraic form of the Hamiltonian (4.7). In the rational case the relevant variables were the polynomials generating the S^W algebra of the Weyl-invariant polynomials of x_i . This algebra allows the grading by means of homogeneous polynomials. In the trigonometric case we are going to deal with Weyl-invariant symmetric *trigonometric* polynomials τ_k in variables

$$y_i = \frac{\sin(\beta x_i)}{\beta} , \quad (4.8)$$

which can be represented as polynomials in

$$\sigma_k(x) = S_k(y_i^2) , \quad (4.9)$$

where S_k are elementary symmetric polynomials.

We impose an important requirement of a correspondence (the ‘correspondence’ principle, see above) that in the limit $\beta \rightarrow 0$ the new variables should coincide with the variables (3.10) found for the rational F_4 model. Therefore, let us define the Weyl-invariant trigonometric polynomials by averaging the elementary trigonometric polynomials over an orbit Ω generated by the root $e_1 + e_2$

$$\tau_a(x, \beta) = \sum_{\alpha \in \Omega} \left(\frac{\sin(\beta(\alpha, x))}{\beta} \right)^{2a} , \quad a = 1, 3, 4, 6 , \quad (4.10)$$

which reproduce the expression (3.8) in the limit $\beta \rightarrow 0$. To simplify formulas one can use the generalized transformations of type (3.9)

$$\tau_a \rightarrow \tau_a + q_a(\tau; \beta) , \quad a = 1, 3, 4, 6 , \quad (4.11)$$

where $q_a(\tau; \beta)$ are polynomials in τ ’s with β -depending coefficients of dimensions $(2a)^{13}$. Finally, we get

$$\tau_1 = \sigma_1 - \frac{2\beta^2}{3}\sigma_2 ,$$

¹³The dimension of τ_a is equal to $(2a)$ while for β it is equal to (-1) .

$$\begin{aligned}
\tau_3 &= \sigma_3 - \frac{1}{6} \sigma_1 \sigma_2 - 2\beta^2 \left(\sigma_4 - \frac{1}{36} \sigma_2^2 \right) , \\
\tau_4 &= \sigma_4 - \frac{1}{4} \sigma_1 \sigma_3 + \frac{1}{12} \sigma_2^2 , \\
\tau_6 &= \sigma_4 \sigma_2 - \frac{1}{36} \sigma_2^3 - \frac{3}{8} \sigma_3^2 + \frac{1}{8} \sigma_1 \sigma_2 \sigma_3 - \frac{3}{8} \sigma_1^2 \sigma_4 .
\end{aligned} \tag{4.12}$$

Quite surprisingly, the variables $\tau_{4,6}$ contain no explicit dependence on β . By construction in the limit $\beta \rightarrow 0$ the variables τ coincide with the variables t of (3.10). The polynomials (4.12) are algebraically independent and generate a certain algebra $S^W(\beta)$ of Weyl-invariant trigonometric polynomials. It is quite clear that $S^W(\beta)$ is isomorphic to the algebra S^W . It simply implies that the algebra S^W has β -parametric realization (4.12).

Finally, the gauge-rotated operator (4.7) in the coordinates (4.12) has an algebraic form

$$h_{F^4}^{(t)} = \sum_{a,b} A_{ab} \frac{\partial^2}{\partial \tau_a \partial \tau_b} + \sum_{a=1} (B_a + C_a) \frac{\partial}{\partial \tau_a} , \quad a, b = 1, 3, 4, 6 , \tag{4.13}$$

where the coefficient functions are

$$\begin{aligned}
A_{11} &= 4 \tau_1 - 4\beta^2 \tau_1^2 - \frac{32}{3} \beta^4 \tau_3 - \frac{128}{9} \beta^6 \tau_4 , \\
A_{13} &= 12 \tau_3 - \frac{8}{3} \beta^2 (4\tau_1 \tau_3 + \tau_4) - \frac{32}{9} \beta^4 \tau_1 \tau_4 , \\
A_{14} &= 16 \tau_4 - \frac{40}{3} \beta^2 \tau_1 \tau_4 - \frac{16}{3} \beta^4 \tau_6 , \\
A_{16} &= 24 \tau_6 - 20\beta^2 \tau_1 \tau_6 - \frac{32}{3} \beta^4 \tau_4^2 , \\
A_{33} &= -\frac{2}{3} \tau_1^2 \tau_3 + \frac{20}{3} \tau_1 \tau_4 - \frac{8}{9} \beta^2 (18\tau_3^2 + \tau_1^2 \tau_4 + 12\tau_6) , \\
A_{34} &= -\frac{4}{3} \tau_1^2 \tau_4 + 8 \tau_6 - \frac{4}{3} \beta^2 (\tau_1 \tau_6 + 12\tau_3 \tau_4) , \\
A_{36} &= 16 \tau_4^2 - 2 \tau_1^2 \tau_6 - \frac{8}{3} \beta^2 (9\tau_3 \tau_6 + \tau_1 \tau_4^2) , \\
A_{44} &= -4 \tau_3 \tau_4 - 2 \tau_1 \tau_6 - 24\beta^2 \tau_4^2 , \\
A_{46} &= -4 \tau_1 \tau_4^2 - 6 \tau_3 \tau_6 - 36\beta^2 \tau_4 \tau_6 , \\
A_{66} &= -12\tau_3 \tau_4^2 - 6\tau_1 \tau_4 \tau_6 - 8\beta^2 (6\tau_6^2 + \tau_4^3) ,
\end{aligned}$$

$$A_{ba} = A_{ab} , \quad (4.14)$$

and

$$\begin{aligned} B_1 &= 8 - 8\beta^2\tau_1 , & B_3 &= -\tau_1^2 - \frac{56}{3}\beta^2\tau_3 - \frac{32}{9}\beta^4\tau_4 , \\ B_4 &= -4\tau_3 - \frac{88}{3}\beta^2\tau_4 , & B_6 &= -8\tau_1\tau_4 - 56\beta^2\tau_6 . \end{aligned} \quad (4.15)$$

The coefficients A_{ab} have a meaning of elements of a metric with upper indexes which corresponds to the flat space for any value of the parameter β . Hence, the operator (4.13) with the coefficients A_{ab} and B_a (when $C_a = 0$) defines the flat space Laplacian in an *algebraic* form. At $\beta \rightarrow 0$ the expressions (4.14)–(4.15) become (3.11)–(3.12). Terms stemming from the potential part of the Hamiltonian are proportional to μ, ν

$$\begin{aligned} C_1 &= 48(\nu + \mu) - 8\beta^2(5\nu + 6\mu)\tau_1 , & C_3 &= -2(2\nu + \mu)\tau_1^2 - 16\beta^2(3\nu + 5\mu)\tau_3 , \\ C_4 &= -12\nu\tau_3 - 24\beta^2(3\nu + 4\mu)\tau_4 , & C_6 &= -12\nu\tau_1\tau_4 - 48\beta^2(2\nu + 3\mu)\tau_6 , \end{aligned} \quad (4.16)$$

(cf. (3.13)).

Eventually, the operator (4.13) with the coefficients (4.14)–(4.16) presents an *algebraic* form of the F_4 trigonometric model. It is straightforward to check that the operator (4.13) with the coefficients A_{ab} , B_a , C_a from (4.14)–(4.16) preserves the *same* flag of spaces of polynomials $\mathcal{P}^{(F_4)}$ (3.16) as in the rational case (see (3.11))¹⁴. Thus, it is evident that the F_4 trigonometric Hamiltonian in the algebraic form (4.13) can be rewritten in terms of the generators of the $f^{(4)}$ -algebra.

4.2 Duality

Let us make now a short digression on a parallel between two equivalent representations of the F_4 Hamiltonian related to two dual root systems mentioned in Introduction. These representations are connected by the ‘duality’ transformation \mathcal{D} , Eq.(1.8),

$$z = \mathcal{D}x \quad , \quad \mathcal{D}^{-1} = \frac{1}{2}\mathcal{D} ,$$

and we shall refer to two forms of Hamiltonian as to z - and x -representations, correspondingly. The functional form of the Hamiltonian in these representations is remarkably similar (but do not coincide) because of similarity of two root systems. For the rational

¹⁴Note that according to the papers [10, 17] the rational and trigonometric F_4 Hamiltonians preserve different minimal flags. Neither of them correspond to the flag $\mathcal{P}^{(F_4)}$ (3.16).

case the only change occurs in values of coupling constants g and g_1 while for the trigonometric case a rescaling of the parameter β has to be done for some terms in the potential

The dual transformation (1.8) being applied to the Hamiltonian (4.1) provides the following correspondence:

$$\begin{aligned} \sum_{i=1}^4 \partial_{x_i}^2 &\rightleftharpoons 2 \sum_{i=1}^4 \partial_{z_i}^2 , \\ V_1(x, \beta) &\rightleftharpoons V_2(z, \beta) , \\ V_2(x, 2\beta) &\rightleftharpoons 4V_1(z, \beta) . \end{aligned} \quad (4.17)$$

It implies that after the substitution (1.8) we arrive at the equivalent Hamiltonian in the Olshanetsky–Perelomov form [2]

$$H_{F_4}^{(OP)}(z) = -\frac{1}{2} \sum_{i=1}^4 \partial_{z_i}^2 + \mu(\mu-1)V_1(z, \beta) + \frac{\nu(\nu-1)}{2}V_2(z, \beta) , \quad (4.18)$$

with evident rescaling of the spectrum

$$E \rightleftharpoons E/2 . \quad (4.19)$$

The factors of the wave function (4.4) corresponding to the potentials V_1 and V_2 also satisfy the 'duality' relations:

$$\begin{aligned} (\Delta_+ \Delta_-)(x, \beta) &= (\Delta_0 \Delta)(z, \beta) , \\ (\Delta_0 \Delta)(x, 2\beta) &= (\Delta_+ \Delta_-)(z, \beta) . \end{aligned} \quad (4.20)$$

At $\mu = 0$ (corresp. $\nu = 0$) the F_4 Hamiltonian reduces to the D_4 Hamiltonians in x - (corresp. z -) representation. The relevant variables providing an algebraic form to the D_4 Hamiltonian are $\sigma_k(x, \beta)$ — symmetric polynomials in $y_i^2 = \sin^2(\beta x_i)/\beta^2$ (see section 2.2, Eqs.(2.19) - (2.20)). The new variables $\tau_a(x, \beta)$ being polynomials in $\sigma(x, \beta)$ give an algebraic form both to ν - and to μ -parts of the F_4 Hamiltonian. This means that one can expect existence of algebraic relations between dual variables $\tau_a(z, \beta)$ and $\tau_a(x, \beta)$. Indeed, there exist algebraic relations which express the variables $\tau_a(z, \beta)$ through the $\tau_a(x, \beta)$:

$$\tau_a(z, \beta) = p_a(\tau(x, \beta); \beta) \quad a = 1, 3, 4, 6 , \quad (4.21)$$

where $p_a(\tau; \beta)$ are polynomials in τ 's with β -depending coefficients of the following form

$$p_1(\tau; \beta) = 2\tau_1 - \frac{8}{3}\beta^2\tau_1^2 + \frac{256}{3}\beta^4\tau_3 ,$$

$$\begin{aligned}
p_3(\tau; \beta) &= -8\tau_3 - \frac{1}{3}\tau_1^3 + 4\beta^2 \left(-16\tau_4 + \frac{1}{18}\tau_1^4 + \frac{8}{3}\tau_1\tau_3 \right) - \frac{128}{9}\beta^4\tau_1^2\tau_3 + \frac{2048}{9}\beta^6\tau_3^2, \\
p_4(\tau; \beta) &= 16\tau_4 + 4\tau_1\tau_3 + \frac{1}{12}\tau_1^4 - 4\beta^2 \left(2\tau_1\tau_4 + \frac{1}{3}\tau_1^2\tau_3 \right) + \frac{16}{3}\beta^4(2\tau_6 + \tau_3^2), \\
p_6(\tau; \beta) &= -64\tau_6 - 24\tau_3^2 - 8\tau_1^2\tau_4 - 2\tau_1^3\tau_3 - \frac{1}{36}\tau_1^6 \\
&\quad - 4\beta^2(8\tau_3\tau_4 - 16\tau_1\tau_6 - 4\tau_1\tau_3^2 - \tau_1^3\tau_4 - \frac{1}{6}\tau_1^4\tau_3) \\
&\quad - 16\beta^4(6\tau_4^2 + 2\tau_1\tau_3\tau_4 + \frac{1}{3}\tau_1^2\tau_6 + \frac{1}{3}\tau_1^2\tau_3^2) + \frac{32}{3}\beta^6(\tau_3\tau_6 + \frac{1}{3}\tau_3^3). \tag{4.22}
\end{aligned}$$

The inverse relations for $\tau(x, \beta)$'s as functions of $\tau(z, \beta)$'s are not algebraic. However it is possible to express algebraically $\tau(x, \beta)$'s through the $\tau(z, \beta/2)$'s with the use of eqs (4.21):

$$\tau_a(x, \beta) = p_a(\tau(\mathcal{D}^{-1}x, \beta); \beta) = p_a(\tau(z/2, \beta); \beta). \tag{4.23}$$

The relations (4.23) suggest another way of finding variables in which the F_4 model acquires an algebraic form — one has to look for algebraic relations between $\sigma_k = S_k(\sin^2(\beta x_i)/\beta^2)$ and $\check{\sigma}_k = S_k(\sin^2(\beta z_i/2)/\beta^2)$:

$$f_a(\sigma; \beta) = g_a(\check{\sigma}; \beta). \tag{4.24}$$

Polynomial functions f_a and g_a can be used as new variables which give an algebraic form to the Hamiltonian in x and z representations, correspondingly. Historically we used namely relations (4.24) for finding relevant variables (4.12):

$$\tau_a \equiv f_a(\sigma; \beta).$$

Explicit formulas for $g_a(\check{\sigma}; \beta)$ are given in Appendix C. One can see that the functions f_a have much simpler form than the functions g_a what gives a preference to the x -representation.

4.3 Wave functions and energies of the trigonometric F_4 model

Configuration space of the trigonometric F_4 model is defined by zeros of the ground state eigenfunction (4.4), i.e. of the expression $(\Delta_+ \Delta_- \Delta_0 \Delta)^2$. This expression can be written as a product of two factors, (i):

$$(\Delta_+ \Delta_-)^2 = 256\tau_4^3 - 192\tau_6^2, \tag{4.25}$$

which corresponds to the trigonometric D_4 model appearing at $g_1 = 0$, and (ii): $(\Delta_0 \Delta)^2$, which corresponds to the degenerate ($g = 0$) F_4 model (cf. (3.19), (3.20)). In order to

find the second factor we use the second relation (4.20). It tells that this factor is equal to $\Delta_+^2 \Delta_-^2$ written in z -variables and, thus, it can be expressed through $\tau_{4,6}(z, \beta)$,

$$(\Delta \Delta_0)^2 = 256\tau_4(z, \beta)^3 - 192\tau_6(z, \beta)^2. \quad (4.26)$$

The use of eqs (4.21), (4.22) gives

$$\begin{aligned} \Delta_0^2 \Delta^2 = & \left(-\frac{8}{3}\tau_1^6\tau_6 + 8\tau_1^5\tau_3\tau_4 + 16\tau_1^4\tau_4^2 - 8\tau_1^3\tau_3^3 - 192\tau_1^3\tau_3\tau_6 + 480\tau_1^2\tau_3^2\tau_4 \right. \\ & - 768\tau_1^2\tau_4\tau_6 + 3072\tau_1\tau_3\tau_4^2 - 432\tau_3^4 - 2304\tau_3^2\tau_6 + 4096\tau_4^3 - 3072\tau_6^2) \\ & + \frac{8}{3}\beta^2(\tau_1^7\tau_6 - 3\tau_1^6\tau_3\tau_4 - 6\tau_1^5\tau_4^2 + 3\tau_1^4\tau_3^3 + 96\tau_1^4\tau_3\tau_6 - 246\tau_1^3\tau_3^2\tau_4 \\ & + 288\tau_1^3\tau_4\tau_6 - 1120\tau_1^2\tau_3\tau_4^2 + 144\tau_1\tau_3^4 - 1536\tau_1\tau_4^3 + 960\tau_1\tau_3^2\tau_6 + 1536\tau_1\tau_6^2 \\ & - 288\tau_3^3\tau_4 - 768\tau_3\tau_4\tau_6) + \frac{16}{3}\beta^4(246\tau_1^3\tau_3\tau_4^2 + 768\tau_1\tau_3\tau_4\tau_6 + 312\tau_1\tau_3^3\tau_4 \\ & - 324\tau_1^2\tau_3^2\tau_6 - 24\tau_3^2\tau_4^2 - 68\tau_1^4\tau_4\tau_6 + 30\tau_1^4\tau_3^2\tau_4 - 192\tau_4^2\tau_6 - 11\tau_1^5\tau_3\tau_6 \\ & - 672\tau_1^2\tau_6^2 + 360\tau_1^2\tau_4^3 - 24\tau_1^2\tau_3^4) - \frac{32}{3}\beta^6(\tau_1^5\tau_4\tau_6 - 3\tau_1^4\tau_3\tau_4^2 - 40\tau_1^3\tau_3^2\tau_6 \\ & - 6\tau_1^3\tau_4^3 - 48\tau_1^3\tau_6^2 + 96\tau_1^2\tau_3^3\tau_4 + 8\tau_1^2\tau_3\tau_4\tau_6 + 312\tau_1\tau_3^2\tau_4^2 - 96\tau_1\tau_4^2\tau_6 - 48\tau_3^5 \\ & - 272\tau_3^3\tau_6 + 432\tau_3\tau_4^3 - 384\tau_3\tau_6^2) + \frac{64}{9}\beta^8(\tau_1^4\tau_6^2 + 24\tau_1^3\tau_3\tau_4\tau_6 - 72\tau_1^2\tau_3^2\tau_4^2 \\ & + 180\tau_1^2\tau_4^2\tau_6 - 648\tau_1\tau_3\tau_4^3 - 144\tau_1\tau_3^3\tau_6 - 384\tau_1\tau_3\tau_6^2 + 288\tau_3^4\tau_4 + 1056\tau_3^2\tau_4\tau_6 \\ & - 972\tau_4^4 + 768\tau_4\tau_6^2) - \frac{1024}{9}\beta^{10}(\tau_1^2\tau_3\tau_6^2 + 6\tau_1\tau_3^2\tau_4\tau_6 + 24\tau_1\tau_4\tau_6^2 - 18\tau_3^3\tau_4^2 \\ & - 54\tau_3\tau_4^2\tau_6) + \frac{4096}{27}\beta^{12}(3\tau_3^2 + 8\tau_6)\tau_6^2, \end{aligned} \quad (4.27)$$

(cf. (3.20)), which corresponds to the degenerate F_4 model at $g = 0$. A boundary of the configuration space of the trigonometric F_4 model is confined by the algebraic surfaces (4.25), (4.27) of the third and eighth orders, correspondingly. It is quite surprising that in the rational case the corresponding surface defined by (3.20) is of the seventh order while in (4.27) there exist the only two terms of the eighth order: $(8/3)\beta^2(\tau_1^7\tau_6 - 3\tau_1^6\tau_3\tau_4)$.

It is remarkable that in the F_4 trigonometric case the relation between the Jacobian and the ground-state wave function has the same simple form as in the rational case:

$$\left[\det \left(\frac{\partial \tau_a}{\partial x_k} \right) \right]^2 = \frac{1}{4096} (\Delta_+ \Delta_-)^2 (\Delta_0 \Delta)^2. \quad (4.28)$$

(cf. (3.21)).

Let us now proceed to finding the spectrum of the Hamiltonian (4.13). It is easy to see that the operator (4.13) with the coefficients (4.14)–(4.16) has a block triangular form

in the τ variables unlike pure triangular form which is needed in order to find eigenvalues. In general, in order to reduce this operator to pure triangular form it is necessary to diagonalize each block separately, doing it one by one. Surprisingly, in our particular problem it can be performed in full generality just by introducing unique set of new variables (!)

$$\begin{aligned}
\rho_1 &= \tau_1 , \\
\rho_3 &= \tau_3 - \frac{1}{8}\beta^{-2}\tau_1^2 , \\
\rho_4 &= \tau_4 - \frac{3}{16}\beta^{-4}\tau_1^2 , \\
\rho_6 &= \tau_6 - \frac{3}{4}\beta^{-2}\tau_1\tau_4 + \frac{3}{64}\beta^{-6}\tau_1^3 ,
\end{aligned} \tag{4.29}$$

having the same dimension as in (4.12). It is worth to note that this substitution becomes singular at $\beta = 0$, reflecting the non-existence of bound states for the rational D_4 and F_4 models in absence of the harmonic oscillator term in potential. This coordinate transformation is of the type (4.11) and hence leaves the flag (3.16) invariant. Thus, we arrive at a conclusion that among Weyl-invariant coordinate systems (of minimal dimension) there exists unique one which leads to pure triangular form of the Hamiltonian with respect to a basis of monomials. We were unable to see a relation with variables introduced in [17, 18] in framework of a general study of trigonometric Hamiltonians based on root system which should guarantee triangular form.

In the coordinates (4.29) the Hamiltonian (4.7) takes the form

$$\begin{aligned}
h_{F_4}^{(t)} &= \left(4\rho_1 - 8\beta^2\rho_1^2 - \frac{32}{3}\beta^4\rho_3 - \frac{128}{9}\beta^6\rho_4\right)\partial_{\rho_1}^2 + 8\left[3\rho_3 - 2\beta^2\left(\frac{1}{3}\rho_4 + \rho_3\rho_1\right)\right]\partial_{\rho_1\rho_3}^2 \\
&+ \left(3\beta^{-4}\rho_1^2 + 32\rho_4 + 8\rho_3\rho_1 - 24\beta^2\rho_4\rho_1 - \frac{32}{3}\beta^4\rho_6\right)\partial_{\rho_1\rho_4}^2 + 2\left[3\beta^{-2}\rho_1\rho_4\right. \\
&+ \left.24\rho_6 - 4\beta^2\left(\rho_1\rho_6 - 2\rho_3\rho_4\right)\right]\partial_{\rho_1\rho_6}^2 - \left[6\beta^{-2}\rho_1\rho_3 + 16\beta^2\left(\rho_3^2 + \frac{2}{3}\rho_6\right)\right]\partial_{\rho_3}^2 \\
&+ \left(-9\beta^{-4}\rho_1\rho_3 + 16\rho_6 - 32\beta^2\rho_3\rho_4 + 6\beta^{-2}\rho_1\rho_4\right)\partial_{\rho_3\rho_4}^2 \\
&+ \left[-6\beta^{-2}\left(3\rho_3\rho_4 - 4\rho_1\rho_6\right) + 36\rho_4^2 - 48\beta^2\rho_3\rho_6\right]\partial_{\rho_3\rho_6}^2 \\
&+ \left[-\frac{27}{16}\beta^{-8}\rho_1^3 - \frac{3}{4}\beta^{-4}\rho_1\left(3\rho_1\rho_3 - 16\rho_4\right) + 2\rho_1\rho_6 - 4\rho_3\rho_4 - 24\beta^2\rho_4^2\right]\partial_{\rho_4}^2 \\
&- \left[+\frac{27}{4}\beta^{-6}\rho_1^2\rho_4 + 18\beta^{-4}\rho_1\rho_6 + 3\beta^{-2}\rho_4\left(3\rho_1\rho_3 + 8\rho_4\right) + 12\rho_3\rho_6\right. \\
&+ \left.64\beta^2\rho_4\rho_6\right]\partial_{\rho_4\rho_6}^2 - \left[\frac{27}{4}\beta^{-4}\rho_1\rho_4^2 - 9\beta^{-2}\rho_6\left(\rho_1\rho_3 - 4\rho_4\right) + 18\rho_3\rho_4^2 + 48\beta^2\rho_6^2\right]\partial_{\rho_6}^2
\end{aligned}$$

$$\begin{aligned}
& + 8 \left[(1 + 6\nu + 6\mu) - \beta^2 \rho_1 (1 - 6\mu - 5\nu) \right] \partial_{\rho_1} - \left[3\beta^{-2} (1 + 4\nu + 4\mu) \rho_1 \right. \\
& + 16\beta^2 (1 + 3\nu + 5\mu) \rho_3 \left. \right] \partial_{\rho_3} - \left[\frac{9}{2} \beta^{-4} (1 + 4\nu + 4\mu) \rho_1 + 12\nu \rho_3 \right. \\
& + 24\beta^2 (1 + 3\nu + 4\mu) \rho_4 \left. \right] \partial_{\rho_4} - \left[\frac{27}{8} \beta^{-6} \rho_1^2 - 9\beta^{-4} \nu \rho_1 \rho_3 + 6\beta^{-2} (5 + 6\nu \right. \\
& + 6\mu) \rho_4 + 48\beta^2 (1 + 2\nu + 3\mu) \rho_6 \left. \right] \partial_{\rho_6} , \tag{4.30}
\end{aligned}$$

and it seems simpler than the operator (4.13) with the coefficients (4.14)–(4.16). It is easy to check that the operator (4.30) is indeed a triangular operator. It is evident that this operator can be rewritten in terms of the $f^{(4)}$ -generators like it was for the operator (4.13).

Using the representation (4.30) the energy levels of the Hamiltonian, $h_{F_4}^{(t)} \varphi = -2\epsilon \varphi$, can be found explicitly and are given by

$$\begin{aligned}
\epsilon_n = & 4\beta^2 [p_1(p_1 + 2p_3 + 3p_4 + 4p_6) + 2p_3(p_3 + 2p_4 + 3p_6) + p_4(3p_4 + 8p_6) \\
& + 6p_6^2 + \nu(5p_1 + 6p_3 + 9p_4 + 12p_6) + 2\mu(3p_1 + 5p_3 + 6p_4 + 9p_6)] \tag{4.31}
\end{aligned}$$

where $n = 0, 1, \dots$, and quantum numbers p_a are non-negative integers with a condition $(p_1 + 2p_3 + 2p_4 + 3p_6) = n$. The spectrum of the original trigonometric F_4 Hamiltonian (4.1) is $E_n = E_0 + \epsilon_n$ (cf. (2.23)).

The explicit expressions for the first several eigenfunctions of (4.30) in ρ -variables are presented in Appendix D. It is worth to mention that the equations describing the boundary of the configuration space remain algebraic in ρ -variables and are given by

$$(\Delta_+ \Delta_-)^2 = -36\beta^{-6} \rho_1^3 \rho_6 + 36\beta^{-4} \rho_1^2 \rho_4^2 - 288\beta^{-2} \rho_1 \rho_4 \rho_6 + 256\rho_4^3 - 192\rho_6^2 , \tag{4.32}$$

and

$$\begin{aligned}
(\Delta \Delta_0)^2 = & -72\beta^{-6} \rho_1^3 (3\rho_3^2 + 8\rho_6) + 36\beta^{-4} \rho_1^2 (24\rho_1^2 \rho_6 + 9\rho_1^2 \rho_3^2 + 16\rho_4^2) \\
& - 144\beta^{-2} \rho_1 (12\rho_3^2 \rho_4 + 8\rho_3 \rho_1 \rho_6 + 32\rho_6 \rho_4 + 3\rho_3^3 \rho_1 + 6\rho_1^2 \rho_4^2) \\
& - (3072\rho_6^2 + 4096\rho_4^3 + 864\rho_1^3 \rho_3^3 + 2880\rho_4 \rho_1^2 \rho_3^2 + 2304\rho_6 \rho_1^3 \rho_3 \\
& + 768\rho_3 \rho_4^2 \rho_1 + 7680\rho_4 \rho_1^2 \rho_6 - 2304\rho_3^2 \rho_6 - 432\rho_3^4) \\
& - 192\beta^2 (9\rho_1^2 \rho_3 \rho_4^2 - 32\rho_1 \rho_6^2 - 3\rho_1 \rho_3^4 - 20\rho_1 \rho_3^2 \rho_6 + 6\rho_4 \rho_3^3 \\
& + 36\rho_4^3 \rho_1 + 16\rho_3 \rho_4 \rho_6) + 64\beta^4 (60\rho_1 \rho_4 \rho_3^3 + 12\rho_1^2 \rho_3^2 \rho_6 - 2\rho_4^2 \rho_3^2 \\
& + 9\rho_1^2 \rho_3^4 - 16\rho_4^2 \rho_6 - 32\rho_1^2 \rho_6^2 + 160\rho_1 \rho_3 \rho_4 \rho_6) + \frac{256}{3} \beta^6 (-54\rho_4^3 \rho_3
\end{aligned}$$

$$\begin{aligned}
& + 108 \rho_4^2 \rho_6 \rho_1 + 27 \rho_3^2 \rho_4^2 \rho_1 + 34 \rho_6 \rho_3^3 + 48 \rho_6^2 \rho_3 + 6 \rho_3^5 \Big) \\
& - \frac{256}{3} \beta^8 \Big(- 88 \rho_3^2 \rho_4 \rho_6 - 64 \rho_4 \rho_6^2 + 81 \rho_4^4 + 32 \rho_1 \rho_6^2 \rho_3 - 24 \rho_3^4 \rho_4 \\
& + 12 \rho_1 \rho_6 \rho_3^3 \Big) + 2048 \beta^{10} \rho_3 \rho_4^2 \Big(\rho_3^2 + 3 \rho_6 \Big) + \frac{4096}{27} \beta^{12} \rho_6^2 \Big(3 \rho_3^2 + 8 \rho_6 \Big) .
\end{aligned} \tag{4.33}$$

Now the boundary of the configuration space of the trigonometric F_4 model is confined by the algebraic surfaces (4.32)–(4.33) of the fourth and sixth orders, correspondingly,¹⁵ unlike the τ -variables where they were of the third and eighth orders, respectively (cf. (4.32)–(4.33)).

5 Conclusion

We have found that the general rational and trigonometric F_4 integrable models with two arbitrary coupling constants are exactly-solvable. After gauging away the ground state eigenfunction these models look very much alike when written in a certain Weyl-invariant variables. Their Hamiltonians preserve the same flag of the spaces of polynomials and both models are characterized by the same hidden algebra – they are a non-linear combination in generators. It is an infinite-dimensional but finite-generated Lie algebra of the differential operators, which we call $f^{(4)}$ algebra. It is quite interesting that D_4 rational and trigonometric models possess *two* hidden algebras: $gl(5)$ and $f^{(4)}$. Similar situation takes place for A_2 , BC_2 rational and trigonometric models as well as for G_2 rational model. Their hidden algebras were $gl(3)$ and $g^{(2)}$, respectively, however, G_2 trigonometric model had the only hidden algebra $g^{(2)}$ (see [9]).

Present work complements previous studies where the algebraic and the Lie-algebraic forms as well as the corresponding flags were found for the rational and trigonometric Olshanetsky-Perelomov Hamiltonians of $ABCD$ series (and their supersymmetric generalizations) [5, 7] and G_2 model [8]. In order to conclude a study the whole set of Olshanetsky-Perelomov integrable systems appearing in the Hamiltonian reduction method it is necessary to perform the same analysis for remaining $E_{6,7,8}$ integrable rational and trigonometric models. We consider it as a challenging task for future.

Our concrete consideration does not confirm some results presented in [10, 17] for the rational and trigonometric F_4 models. In particular, we found that the minimal flags of spaces of polynomials preserved by the rational and trigonometric F_4 Hamiltonians coincide, however, for both cases they differ from those given in [10, 17]. For trigonometric

¹⁵Being in total the algebraic surface of the tenth order.

case our Weyl-invariant variables leading to pure triangular form of the F_4 Hamiltonian are singular in β . It reflects the fact that the triangular form for rational and trigonometric models has a different origin: for the rational case the diagonal matrix elements are proportional to ω while for trigonometric one to β . The variables (4.29) look different from those presented in a general scheme [17, 18] formulated for all root systems.

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A Representation of the algebra $gl(5)$

The algebra $gl(5)$ has a realization in terms of first order differential operators in four-dimensional (s_1, s_2, s_3, s_4) -space:

$$\begin{aligned} J_i^- &= \partial_i, \quad i = 1, 2, 3, 4, \\ J_{ik}^0 &= s_i \partial_k, \quad i, k = 1, 2, 3, 4, \\ J^0 &= n - \sum s_i \partial_i, \\ J_i^+ &= s_i J^0, \quad i = 1, 2, 3, 4, \end{aligned} \tag{A.1}$$

where $n \in \mathbb{C}$ is a free parameter. In this realization a grading $(a_i | i = 1, 2, 3, 4)$ can be assigned to the generators (A.1) through their action on monomial

$$J s_1^{p_1} s_2^{p_2} s_3^{p_3} s_4^{p_4} \propto s_1^{p_1+a_1} s_2^{p_2+a_2} s_3^{p_3+a_3} s_4^{p_4+a_4}.$$

Then the grading of J is defined as a four-component vector $\vec{a} = (a_1, a_2, a_3, a_4)$.

If n is a non-negative integer, the representation (A.1) becomes finite-dimensional and the corresponding representation space is a linear space of polynomials

$$\mathcal{P}_n = \langle s_1^{p_1} s_2^{p_2} s_3^{p_3} s_4^{p_4} | 0 \leq (p_1 + p_2 + p_3 + p_4) \leq n \rangle. \tag{A.2}$$

For fixed n the algebra (A.1) acts on the space (A.2) irreducibly. As a function of n the spaces \mathcal{P}_n possess a property that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{Z}_+$ and form an infinite flag (filtration)

$$\bigcup_{n \in \mathbb{Z}_+} \mathcal{P}_n = \mathcal{P}.$$

B The $f^{(4)}$ algebra

We define the algebra $f^{(4)}$ as an algebra of differential operators on \mathbb{C}^4 which acts irreducibly on the space of inhomogeneous polynomials in four variables

$$\mathcal{P}_n = \langle s_1^{p_1} s_2^{p_2} s_3^{p_3} s_4^{p_4} | 0 \leq p_1 + 2p_2 + 2p_3 + 3p_4 \leq n \rangle, \tag{B.1}$$

where $n \in \mathbb{N}$. Thus, $f^{(4)} \subset \text{diff}(\mathbb{C}^4)$.

The structure of the algebra $f^{(4)}$ is the following. It contains three abelian subalgebras $R^{(k)}$, $k = 2, 3, 4$ of first order differential operators

$$\begin{aligned} R_i^{(2)} &= s_1^i \partial_2, \quad i = 0, 1, 2, \\ R_i^{(3)} &= s_1^i \partial_3, \quad i = 0, 1, 2, \\ R_i^{(4)} &= s_1^i \partial_4, \quad i = 0, 1, 2, 3, \end{aligned} \tag{B.2}$$

and 11-dimensional subalgebra B of first order differential operators

$$\begin{aligned} B_0^{(1)} &= \partial_1, & B_1^{(1)} &= s_1 \partial_1, & B_2^{(2)} &= s_2 \partial_2, & B_3^{(2)} &= s_3 \partial_2, \\ B_2^{(3)} &= s_2 \partial_3, & B_3^{(3)} &= s_3 \partial_3, & B_2^{(4)} &= s_2 \partial_4, & B_3^{(4)} &= s_3 \partial_4, \\ B_4^{(4)} &= s_4 \partial_4, & B_{12}^{(4)} &= s_1 s_2 \partial_4, & B_{13}^{(4)} &= s_1 s_3 \partial_4. \end{aligned} \quad (\text{B.3})$$

These algebras form a subalgebra $B \ltimes (R^{(2)} \oplus R^{(3)} \oplus R^{(4)}) \subset f^{(4)}$. Also there exists a set of the second order differential operators

$$\begin{aligned} T_2^{(11)} &= s_2 \partial_{11}^2, & T_3^{(11)} &= s_3 \partial_{11}^2, & T_4^{(12)} &= s_4 \partial_{12}^2, & T_4^{(13)} &= s_4 \partial_{13}^2, \\ T_4^{(22)} &= s_4 \partial_{22}^2, & T_{14}^{(22)} &= s_1 s_4 \partial_{22}^2, & T_4^{(23)} &= s_4 \partial_{23}^2, & T_{14}^{(23)} &= s_1 s_4 \partial_{23}^2, \\ T_4^{(33)} &= s_4 \partial_{33}^2, & T_{14}^{(33)} &= s_1 s_4 \partial_{33}^2, & T_{22}^{(14)} &= s_2^2 \partial_{14}^2, & T_{23}^{(14)} &= s_2 s_3 \partial_{14}^2, \\ T_{33}^{(14)} &= s_3^2 \partial_{14}^2, & T_{22}^{(44)} &= s_2^3 \partial_{44}^2, & T_{223}^{(44)} &= s_2^2 s_3 \partial_{44}^2, & T_{233}^{(44)} &= s_2 s_3^2 \partial_{44}^2, \\ T_{333}^{(44)} &= s_3^3 \partial_{44}^2, \end{aligned} \quad (\text{B.4})$$

and third order differential operators

$$\begin{aligned} T_4^{(111)} &= s_4 \partial_{111}^3, & T_{44}^{(222)} &= s_4^2 \partial_{222}^3, & T_{44}^{(223)} &= s_4^2 \partial_{223}^3, & T_{44}^{(233)} &= s_4^2 \partial_{233}^3, \\ T_{44}^{(333)} &= s_4^2 \partial_{333}^3. \end{aligned} \quad (\text{B.5})$$

One can show that the infinite-dimensional algebra generated by 43 generators R, B, T possesses infinitely many common invariant subspaces: it leaves invariant the space \mathcal{P}_n for any $n \in \mathcal{N}$ and therefore preserves the flag (3.16).

Let us introduce an auxiliary generator

$$J^0 = s_1 \partial_1 + 2s_2 \partial_2 + 2s_3 \partial_3 + 3s_4 \partial_4 - n.$$

Then one can define six ‘raising’ generators

$$\begin{aligned} J_1^+ &= s_1 J^0, & J_{3,-2}^+ &= s_3 \partial_2 J^0, & J_{4,-2}^+ &= s_4 \partial_2 J^0, \\ J_{22,-4}^+ &= s_2^2 \partial_4 J^0, & J_{23,-4}^+ &= s_2 s_3 \partial_4 J^0, & J_{22,-2}^+ &= s_2^2 \partial_4 J^0. \end{aligned} \quad (\text{B.6})$$

Finally, we get the infinite-dimensional algebra generated by 49 generators (B.2) – (B.6) and it is called by definition $f^{(4)}$.

These raising generators (B.6) determine a highest-weight vector if n is a non-negative integer number. It leads to a finite-dimensional representation. It can be demonstrated that the operators (B.2) – (B.6) leave the space (B.1) invariant at fixed n and act on it irreducibly. Hence, as stated by the Burnside theorem [19], any operator acting on (B.1) allows a representation as a non-linear combination of the operators (B.2) – (B.6) plus an operator annihilating (B.1) (annihilator). Therefore, the endomorphism of the space (B.1) is given by the infinite-dimensional algebra $f^{(4)}$ generated by 49 generators (B.2) – (B.6). In turn, subalgebra of $f^{(4)}$ generated by the generators (B.2) – (B.5) possesses infinitely many finite-dimensional invariant subspaces (B.1) at $n = 0, 1, \dots$ and hence preserve the infinite non-classical flag of spaces of polynomials (3.16), $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots \subset \mathcal{P}_n \subset \dots$.

C Dual relations

In this Appendix we present polynomial functions $g_a(\check{\sigma}; \beta)$ which satisfy the eq.(4.24) at $\check{\sigma}_k = S_k(\sin^2(\beta z_k/2)/\beta^2)$. These functions can be used as variables providing an algebraic form for the F_4 Hamiltonian in the z -representation (4.18). Compare these formulas with much simpler form of the functions $\tau_a = f_a(\sigma; \beta)$ from eq.(4.12) which give an algebraic form to the Hamiltonian in x -representation.

$$\begin{aligned}
g_1(\check{\sigma}; \beta) &= 2\check{\sigma}_1 - \frac{2}{3}\beta^2(2\check{\sigma}_2 + \check{\sigma}_1^2) + \frac{16}{3}\beta^4\check{\sigma}_3 - \frac{32}{3}\beta^6\check{\sigma}_4, \\
g_3(\check{\sigma}; \beta) &= -8\check{\sigma}_3 + \frac{4}{3}\check{\sigma}_1\check{\sigma}_2 - \frac{1}{3}\check{\sigma}_1^3 + \frac{1}{18}\beta^2(4\check{\sigma}_2\check{\sigma}_1^2 + \check{\sigma}_1^4 - 32\check{\sigma}_2^2 + 120\check{\sigma}_1\check{\sigma}_3) \\
&\quad - \frac{8}{9}\beta^4(6\check{\sigma}_4\check{\sigma}_1 + 2\check{\sigma}_3\check{\sigma}_2 + \check{\sigma}_1^2\check{\sigma}_3) + \frac{16}{9}\beta^6(2\check{\sigma}_4\check{\sigma}_2 + \check{\sigma}_1^2\check{\sigma}_4 + 2\check{\sigma}_3^2) \\
&\quad - \frac{128}{9}\beta^8\check{\sigma}_4\check{\sigma}_3 + \frac{128}{9}\beta^{10}\check{\sigma}_4^2, \\
g_4(\check{\sigma}; \beta) &= 16\check{\sigma}_4 - \frac{2}{3}\check{\sigma}_1^2\check{\sigma}_2 + \frac{4}{3}\check{\sigma}_2^2 + \frac{1}{12}\check{\sigma}_1^4 + \frac{2}{3}\beta^2(\check{\sigma}_1^2\check{\sigma}_3 - 24\check{\sigma}_1\check{\sigma}_4 - 4\check{\sigma}_3\check{\sigma}_2) \\
&\quad + \frac{4}{3}\beta^4(-\check{\sigma}_1^2\check{\sigma}_4 + 16\check{\sigma}_4\check{\sigma}_2 + \check{\sigma}_3^2) - \frac{64}{3}\beta^6\check{\sigma}_4\check{\sigma}_3 + \frac{64}{3}\beta^8\check{\sigma}_4^2, \\
g_6(\check{\sigma}; \beta) &= \frac{16}{9}\check{\sigma}_2^3 - \frac{4}{3}\check{\sigma}_1^2\check{\sigma}_2^2 - \frac{1}{36}\check{\sigma}_1^6 + 16\check{\sigma}_1^2\check{\sigma}_4 - 64\check{\sigma}_4\check{\sigma}_2 + \frac{1}{3}\check{\sigma}_2\check{\sigma}_1^4 \\
&\quad + \frac{1}{3}\beta^2(192\check{\sigma}_4\check{\sigma}_3 + 192\check{\sigma}_4\check{\sigma}_2\check{\sigma}_1 - 48\check{\sigma}_4\check{\sigma}_1^3 - 16\check{\sigma}_3\check{\sigma}_2^2 + 8\check{\sigma}_3\check{\sigma}_2\check{\sigma}_1^2 - \check{\sigma}_3\check{\sigma}_1^4) \\
&\quad + \frac{2}{3}\beta^4(-192\check{\sigma}_4^2 - 96\check{\sigma}_4\check{\sigma}_3\check{\sigma}_1 - 80\check{\sigma}_4\check{\sigma}_2^2 + 16\check{\sigma}_4\check{\sigma}_2\check{\sigma}_1^2 + \check{\sigma}_4\check{\sigma}_1^4 + 8\check{\sigma}_3^2\check{\sigma}_2 \\
&\quad - 2\check{\sigma}_3^2\check{\sigma}_1^2) + \frac{16}{9}\beta^6(72\check{\sigma}_4^2\check{\sigma}_1 + 60\check{\sigma}_4\check{\sigma}_3\check{\sigma}_2 - 6\check{\sigma}_4\check{\sigma}_3\check{\sigma}_1^2 - \check{\sigma}_3^3) \\
&\quad + \frac{32}{3}\beta^8(-16\check{\sigma}_4^2\check{\sigma}_2 + \check{\sigma}_4^2\check{\sigma}_1^2 - 5\check{\sigma}_4\check{\sigma}_3^2) + \frac{512}{3}\beta^{10}\check{\sigma}_4^2\check{\sigma}_3 - \frac{1024}{9}\beta^{12}\check{\sigma}_4^3. \tag{C.7}
\end{aligned}$$

D First eigenfunctions of the rational and trigonometric F_4 models

In this Appendix we present explicit expressions for the first eigenfunctions of F_4 models at $n = 0, 1, 2$.

I. Rational F_4 model.

- $\mathbf{n} = 0$

$$\begin{aligned}\Phi_0 &= 1 , \\ E_0 &= 0 .\end{aligned}$$

• **n = 1**

$$\begin{aligned}\Phi_1 &= s_1 - \frac{2}{\omega}(6\mu + 6\nu + 1) , \\ E_1 &= -4\omega .\end{aligned}$$

• **n = 2**

$$\begin{aligned}\Phi_2^{(1)} &= s_1^2 - \frac{6}{\omega}(4\mu + 4\nu + 1)s_1 + \frac{6}{\omega^2}(4\mu + 4\nu + 1)(6\mu + 6\nu + 1) , \\ E_2^{(1)} &= -8\omega , \\ \Phi_2^{(2)} &= s_3 + \frac{1}{4\omega}(2\mu + 4\nu + 1)s_1^2 - \frac{3}{4\omega^2}(2\mu + 4\nu + 1)(4\mu + 4\nu + 1)s_1 \\ &\quad + \frac{1}{2\omega^3}(2\mu + 4\nu + 1)(6\mu + 6\nu + 1)(4\mu + 4\nu + 1) , \\ E_2^{(2)} &= -12\omega , \\ \Phi_2^{(3)} &= s_4 + \frac{1}{\omega}(3\nu + 1)s_3 + \frac{1}{8\omega^2}(3\nu + 1)(2\mu + 4\nu + 1)s_1^2 \\ &\quad - \frac{1}{4\omega^3}(3\nu + 1)(2\mu + 4\nu + 1)(4\mu + 4\nu + 1)s_1 \\ &\quad + \frac{1}{8\omega^4}(3\nu + 1)(2\mu + 4\nu + 1)(6\mu + 6\nu + 1)(4\mu + 4\nu + 1) , \\ E_2^{(3)} &= -16\omega\end{aligned}$$

II. Trigonometric F_4 model.

• **n = 0**

$$\begin{aligned}\Phi_0 &= 1 , \\ E_0 &= 0 .\end{aligned}$$

• **n = 1**

$$\begin{aligned}\Phi_1 &= \rho_1 - \frac{6\nu + 6\mu + 1}{(5\nu + 6\mu + 1)}\beta^{-2} , \\ E_1 &= 4(5\nu + 6\mu + 1)\beta^2 .\end{aligned}$$

• **n = 2**

$$\begin{aligned}\Phi_2^{(1)} &= \rho_3 + \frac{3}{8} \frac{(4\nu + 4\mu + 1)}{(\nu + 4\mu + 1)}\beta^{-4}\rho_1 - \frac{3}{16} \frac{(4\nu + 4\mu + 1)(6\nu + 6\mu + 1)}{(\nu + 4\mu + 1)(3\nu + 5\mu + 1)}\beta^{-6} , \\ E_2^{(1)} &= 8(3\nu + 5\mu + 1)\beta^2 .\end{aligned}$$

$$\begin{aligned}
\Phi_2^{(2)} &= \frac{2}{3}(3\nu + 2\mu + 1)\beta^2\rho_4 + \nu\rho_3 + \frac{3}{16} \frac{(4\nu + 4\mu + 1)(4\nu + 2\mu + 1)}{(2\nu + 3\mu + 1)}\beta^{-4}\rho_1 \\
&\quad - \frac{1}{16} \frac{(4\nu + 4\mu + 1)(4\nu + 2\mu + 1)(6\nu + 6\mu + 1)}{(2\nu + 3\mu + 1)(3\nu + 4\mu + 1)}\beta^{-6}, \\
E_2^{(2)} &= 12(3\nu + 4\mu + 1)\beta^2. \\
\Phi_2^{(3)} &= \rho_4 + \frac{3}{8} \frac{(3\nu + 1)}{(2\nu + \mu + 1)}\beta^{-2}\rho_3 + \frac{9}{32}(\nu + 1)\beta^{-4}\rho_1^2 \\
&\quad - \frac{9}{64} \frac{(3\nu + 1)(4\nu + 4\mu + 1)(4\nu + 2\mu + 1)}{(5\nu + 6\mu + 3)(2\nu + \mu + 1)}\beta^{-6}\rho_1 \\
&\quad + \frac{9}{128} \frac{(3\nu + 1)(4\nu + 4\mu + 1)(4\nu + 2\mu + 1)(6\nu + 6\mu + 1)}{(5\nu + 6\mu + 3)(2\nu + \mu + 1)(5\nu + 6\mu + 2)}\beta^{-8}, \\
E_2^{(3)} &= 8(5\nu + 6\mu + 2)\beta^2.
\end{aligned}$$

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